SCHAUDER ESTIMATES FOR DEGENERATE MONGE-AMPÈRE EQUATIONS AND SMOOTHNESS OF THE EIGENFUNCTIONS

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ABSTRACT. We obtain $C^{2,\beta}$ estimates up to the boundary for solutions to degenerate Monge-Ampère equations of the type
$$\det D^2 u = f \text{ in } \Omega, \quad f \sim \text{dist}^\alpha(\cdot, \partial\Omega) \text{ near } \partial\Omega, \ \alpha > 0.$$
As a consequence we obtain global C^∞ estimates up to the boundary for the eigenfunctions of the Monge-Ampère operator $(\det D^2 u)^{1/n}$ on smooth, bounded, uniformly convex domains in $\mathbb{R}^n$.

1. INTRODUCTION

In this paper we develop Schauder estimates up to the boundary for degenerate Monge-Ampère equations of the type

$$(1.1) \qquad \det D^2 u = f \text{ in } \Omega, \quad f \sim d_{\partial\Omega}^\alpha \text{ near } \partial\Omega,$$

and apply them to prove global smoothness for the eigenfunctions of the Monge-Ampère operator $(\det D^2 u)^{1/n}$. Throughout the paper, $d_{\partial\Omega}$ represents the distance to the boundary of the domain Ω, and $\alpha > 0$ is a positive power.

Boundary estimates for the Monge-Ampère equation in the nondegenerate case where $f \in C(\overline{\Omega})$ and $f > 0$ were obtained starting with the works of Ivočkina [I], Krylov [K], Caffarelli-Nirenberg-Spruck [CNS] (see also [W]). Also in the nondegenerate case, global $C^{2,\alpha}$ estimates under sharp conditions on the right hand side and boundary data were obtained recently by Trudinger-Wang [TW1] and the second author [S1].

In [S2], the second author established a boundary localization theorem and then applied it to obtain C^2 estimates at the boundary for solutions of (1.1) under natural conditions on the boundary data and the right hand side. In the present work we investigate further regularity of such solutions. We use the boundary C^2 estimates in [S2] and perturbation arguments [C, CC] to establish basic boundary Hölder second derivative estimates for solutions to (1.1) when the boundary $\partial\Omega$, boundary data of u on $\partial\Omega$ and f are more regular; see Theorems 1.1 and 1.2. Our results can be viewed as the degenerate counterparts of the works cited above. They easily give global $C^{2,\beta}$ estimates up to the boundary for the eigenfunctions of the Monge-Ampère operator $(\det D^2 u)^{1/n}$; see Theorem 1.3. This is the key step in settling the open problem on global smoothness of the Monge-Ampère eigenfunctions in all dimensions; see Theorem 1.4.

Before stating our main results we recall the notion for a function to be $C^{2,\beta}$ at a point (see [CC]). We say that u is $C^{2,\beta}$ at x_0 if there exists a quadratic polynomial Q_{x_0} such that, in the domain of definition of u,
$$u(x) = Q_{x_0}(x) + O(|x - x_0|^{2+\beta}).$$
Throughout this paper we refer to a linear map A of the form
$$Ax = x + \tau x_n, \quad \text{with} \quad \tau \cdot e_n = 0,$$
as a *sliding along* $x_n = 0$. Notice that the map A is the identity map when it is restricted to $x_n = 0$ and it becomes a translation of vector $s\tau$ when it is restricted to $x_n = s$.

Let Ω be a bounded convex domain in the upper-half space such that $\partial\Omega$ is $C^{1,1}$ at the origin, that is $0 \in \partial\Omega$ and

(1.2) $$\Omega \subset \{x_n > 0\}, \quad \text{and } \Omega \text{ has an interior tangent ball at the origin.}$$

We are interested in the behavior near the origin of a convex solution $u \in C(\overline{\Omega})$ to the equation

(1.3) $$\det D^2 u = g(x)\, d_{\partial\Omega}^\alpha, \qquad \alpha > 0,$$

where g is a nonnegative function that is continuous at the origin and $g(0) > 0$.

We state below our main Schauder estimates for solutions to (1.3). In Section 2 we will give more precise quantitative versions of these theorems.

Our first result is the following pointwise $C^{2,\beta}$ estimates at the boundary.

Theorem 1.1. *Let Ω, u satisfy (1.2), (1.3) above and let $\beta \in (0, \frac{2}{2+\alpha})$. Assume that*

$$u(0) = 0, \quad \nabla u(0) = 0, \quad u = \varphi \quad on\ \partial\Omega,$$

and the boundary data φ is $C^{2,\beta}$ at 0, and it separates quadratically away from 0.

If $g \in C^\gamma$ at the origin where $\gamma = \beta \frac{2+\alpha}{2}$ then u is $C^{2,\beta}$ at 0. Precisely, there exist a sliding A along $x_n = 0$ and a constant $a > 0$ such that

$$u(Ax) = Q_0(x') + a\, x_n^{2+\alpha} + O(|x'|^2 + x_n^{2+\alpha})^{1+\frac{\beta}{2}},$$

where Q_0 represents the quadratic part of the boundary data φ at the origin.

In fact the conclusion above states that u is pointwise $C^{2,\beta}$ with respect to the degenerate metric specific to our equation (1.3). This is a stronger statement than the pointwise $C^{2,\beta}$ with respect to the Euclidean metric.

In the second theorem we establish the $C^{2,\beta}$ regularity of the solutions to (1.3) in a neighborhood of the origin.

Theorem 1.2. *Let $\beta \in (0, \frac{2}{2+\alpha})$, and $\beta \leq \alpha$. Suppose that u satisfies the equation (1.3) near the origin with*

$$\partial\Omega \in C^{2,\beta}, \quad u|_{\partial\Omega} \in C^{2,\beta}, \quad g \in C^\gamma \quad where \quad \gamma = \beta\,\frac{2+\alpha}{2}.$$

If u separates quadratically from its tangent plane at 0 then

$$u \in C^{2,\beta}(\overline{\Omega} \cap B_\delta(0)) \quad for\ some\ small\ \delta > 0.$$

We remark that the hypothesis $\beta \leq \alpha$ is necessary in the second theorem. This can be easily seen from the expansion given in Theorem 1.1.

As a consequence of our Schauder estimates, we obtain the global $C^{2,\beta}$ regularity, for all $\beta < \frac{2}{2+n}$, of the eigenfunctions to the Monge-Ampère operator $(\det D^2 u)^{1/n}$ on smooth, bounded and uniformly convex domains in $\mathbb{R}^n$. This problem was first investigated by Lions [L]. He showed that there exists a unique (up to positive multiplicative constants) nonzero convex eigenfunction $u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ to the eigenvalue problem for the Monge-Ampère operator $(\det D^2 u)^{1/n}$:

(1.4) $$\begin{cases} (\det D^2 u)^{\frac{1}{n}} = \lambda|u| \text{ in } \Omega, \\ \qquad\quad u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We note that λ, called the Monge-Ampère eigenvalue of Ω, has an interesting stochastic interpretation given by Lions [L]. A variational characterization of λ was found by Tso [T].

The question of global higher derivative estimates up to the boundary of the eigenfunction u is a well known open problem, see for example Trudinger and Wang's survey paper [TW2].

In the two dimensional case, Hong-Huang-Wang in [HHW] resolved this question in the affirmative. In higher dimensions, the global $C^2(\overline{\Omega})$ estimate of the eigenfunction u was obtained by the

second author in [S2]. In Theorem 1.4 below we settle the question of global smoothness of the eigenfunction u in higher dimensions.

Notice that $u \in C^{1,1}(\overline{\Omega})$ implies that we can write $|u| = g\, d_{\partial\Omega}$ with $g \in C^{0,1}(\overline{\Omega})$. Since $u = 0$ on $\partial\Omega$ we also have that u separates quadratically on $\partial\Omega$ from its tangent planes, and therefore we can apply Theorem 1.2.

Theorem 1.3. *Let Ω be a bounded and uniformly convex domain in $\mathbb{R}^n$. Assume $\partial\Omega \in C^{2,\beta}$ with $\beta \in (0, \frac{2}{2+n})$, and u satisfies (1.4). Then $u \in C^{2,\beta}(\overline{\Omega})$.*

Our final theorem shows that the Monge-Ampère eigenfunctions are in fact smooth up to the boundary of Ω if $\partial\Omega$ is smooth. To do this, we first perform a Hodograph transform and then a partial Legendre transform to an eigenfunction u. The linearized operator of the resulting equation can be treated as a degenerate Grushin-type operator. By using the result of Theorem 1.3 and Schauder estimates for Grushin-type operators, we obtain our global smoothness result.

Theorem 1.4. *Let Ω be a bounded and uniformly convex domain in $\mathbb{R}^n$. Assume $\partial\Omega \in C^{\infty}$ and u satisfies (1.4). Then $u \in C^{\infty}(\overline{\Omega})$.*

The paper is organized as follows. In Section 2 we introduce notation and state the quantitative versions of our Schauder estimates. By suitable rescalings we reduce Theorems 1.1 and 1.2 to the Propositions 2.3 and 2.4. These propositions are concerned with the $C^{2,\beta}$ estimates for solutions of (1.3) which are small perturbations of the particular solution U_0 introduced in Section 2. The proofs of Propositions 2.3 and 2.4 will be given in Sections 3 and 4 respectively. In Section 5, we prove Theorem 1.4. Some technical results used in the proofs of the main theorems are collected in the final section, Section 6.

2. Quantitative versions of the Schauder estimates

We introduce some notation. We denote points in $\mathbb{R}^n$ ($n \geq 2$) as

$$x = (x_1, ..., x_n) = (x', x_n), \qquad x' \in \mathbb{R}^{n-1}.$$

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ be the upper half-space. Denote by I and I' the identity matrices of size n and $n - 1$ respectively. We denote by $B_r(x)$ the ball of radius r and center x, and by $B'_r(x')$ the ball in $\mathbb{R}^{n-1}$ of radius r and center x'. For simplicity, we use B_r to denote $B_r(0)$ and $B_r^+ := B_r \cap \{x_n \geq 0\}$.

Given a convex function u defined on a convex set $\overline{\Omega}$, we denote by $S_h(x_0)$ the section centered at x_0 and height $h > 0$,

$$S_h(x_0) := \{x \in \overline{\Omega} | \quad u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\}.$$

We denote for simplicity $S_h = S_h(0)$, and sometimes when specifying the dependence on the function u we use the notation $S_h(u) = S_h$.

Most of the time we write our estimates in the sections of the particular function

$$U_0(x) := \frac{1}{2}|x'|^2 + \frac{1}{(1+\alpha)(2+\alpha)}\, x_n^{2+\alpha}, \quad x \in \mathbb{R}^n_+.$$

We denote by A a sliding along $x_n = 0$, i.e., $Ax = x + \tau x_n$ for some $\tau \in \mathbb{R}^n$ with $\tau \cdot e_n = 0$. Also we denote by F_h the diagonal matrix

$$F_h = \mathrm{diag}(h^{\frac{1}{2}}, h^{\frac{1}{2}}, .., h^{\frac{1}{2}}, h^{\frac{1}{2+\alpha}}).$$

We often rescale solutions in $S_h(u)$ according to the formula

$$u_h(x) = h^{-1}u(AF_hx).$$

For a C^γ continuous function f on a set S, we denote by $[f]_{C^\gamma(S)}$ its homogeneous seminorm, i.e,

$$[f]_{C^\gamma(S)} = \sup_{x,y\in S,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

When $\gamma = 1$, the right hand side is denoted by $[f]_{C^{0,1}(S)}$.

Throughout the paper we think of the constants n, α and β as being fixed. We refer to all positive constants depending only n, α and β as *universal constants* and we denote them by c, C, c_i, C_i. The dependence of various constants also on other parameters like ρ, ρ' will be denoted by $C(\rho, \rho'), c(\rho, \rho')$.

We start by writing quantitative versions of Theorems 1.1 and 1.2.

Theorem 2.1. *Let α, β, γ be as in Theorem 1.1 and ρ, $\rho' \ll \rho$ be small parameters. Let Ω be a bounded convex domain in the upper half-space with $0 \in \partial\Omega$, and assume Ω has an interior ball of radius ρ at the origin. Let $u \in C(\overline{\Omega})$ be convex such that $u(0) = 0, \nabla u(0) = 0$ with*

$$\left| u(x) - \frac{1}{2}|x'|^2 \right| \leq \frac{1}{\rho'}|x'|^{2+\beta} \ \ in \ \partial\Omega \cap B_\rho,$$

and $u \geq \rho'$ on $\partial\Omega \setminus B_\rho$. Assume

$$\det D^2 u = g\, d_{\partial\Omega}^\alpha \quad in \quad \Omega \cap B_\rho, \qquad \det D^2 u \leq \frac{1}{\rho'} \quad in \quad \Omega \setminus B_\rho$$

with

$$|g - 1| \leq \frac{1}{\rho'}|x|^\gamma \ \ in \ \Omega \cap B_\rho.$$

Then there exists a sliding A along $x_n = 0$ with $|A - I| \leq C$ such that

$$|u(Ax) - U_0(x)| \leq C(|x'|^2 + x_n^{2+\alpha})^{1+\frac{\beta}{2}} \ \ in \ B_\rho,$$

with C depending on n, α, β, ρ, ρ'.

We remark that in the proof of Theorem 2.1 we only use the following property of the distance function

$$x_n \geq d_{\partial\Omega}(x) \geq \left(x_n - \rho^{-1}|x'|^2 \right)^+.$$

Theorem 2.2. *Let α, β, γ be as in Theorem 1.2 and ρ, $\rho' \ll \rho$ be small parameters. Assume $\partial\Omega \in C^{2,\beta}$ in B_ρ with $\|\partial\Omega\|_{C^{2,\beta}} \leq 1/\rho'$. Let $u \in C(\overline{\Omega})$ be convex such that*

$$u(0) = 0, \quad \nabla u(0) = 0, \quad u = \varphi(x') \quad on \ \partial\Omega \cap B_\rho,$$

with

$$\|\varphi\|_{C^{2,\beta}(B'_\rho)} \leq \frac{1}{\rho'}, \quad \rho'I' \leq D_{x'}^2\varphi(0) \leq \frac{1}{\rho'}\,I',$$

and $u \geq \rho'$ on $\partial\Omega \setminus B_\rho$. Assume

$$\det D^2 u = g\, d_{\partial\Omega}^\alpha \quad in \quad \Omega \cap B_\rho, \qquad \det D^2 u \leq \frac{1}{\rho'} \quad in \quad \Omega \setminus B_\rho$$

with

$$g \in C^\gamma(\overline{\Omega} \cap B_\rho), \qquad \|g\|_{C^\gamma} \leq \frac{1}{\rho'}, \quad \rho' \leq g(0).$$

Then

$$\|u\|_{C^{2,\beta}(\overline{\Omega} \cap B_\delta)} \leq C,$$

with δ and C depending on n, α, β, ρ, ρ'.

As mentioned in the introduction, the pointwise C^2 estimates for solutions to (1.3) were established in [S2, Theorem 2.4]. This means that after a suitable rescaling (see Lemma 2.5), it suffices to prove our quantitative theorems in the setting when u, and the data g, φ, $\partial\Omega$ are small perturbations of the particular solution U_0,

$$U_0(x) := \frac{1}{2}|x'|^2 + \frac{1}{(1+\alpha)(2+\alpha)} x_n^{2+\alpha},$$

with the data $g \equiv 1$, $\varphi = \frac{1}{2}|x'|^2$, $\partial\Omega = \{x_n = 0\}$. We state below our Theorems 2.1 and 2.2 in this setting.

Let Ω be a convex set with boundary given by the graph of a $C^{1,1}$ convex function $q(x')$, i.e.,

$$\Omega = \{x_n > q(x')\}, \qquad q(0) = 0, \quad \nabla_{x'}q(0) = 0.$$

Let u be a convex function defined in $\overline{\Omega} \cap S_1(U_0)$ that satisfies

(2.1) $$\det D^2 u = g\, d_{\partial\Omega}^\alpha \quad \text{in} \quad \Omega \cap S_1(U_0),$$

(2.2) $$u(0) = 0, \quad \nabla u(0) = 0.$$

and

(2.3) $$u(x) = \varphi(x') \quad \text{on} \quad \partial\Omega.$$

First, we state the pointwise $C^{2,\beta}$ estimates at the origin for solutions to (2.1) which are perturbations of U_0.

Proposition 2.3 (Pointwise $C^{2,\beta}$ estimates). *Assume u satisfies* (2.1), (2.2), (2.3) *with*

$$|u - U_0| \le \varepsilon, \quad |g - 1| \le \varepsilon|x|^\gamma \qquad in \quad \Omega \cap S_1(U_0),$$

with $\gamma = \frac{2+\alpha}{2}$ $\beta \in (0,1)$, and

$$|x_n| \le \varepsilon|x'|^2 \quad and \quad \left|\varphi - \frac{1}{2}|x'|^2\right| \le \varepsilon|x'|^{2+\beta} \qquad on \quad \partial\Omega \cap S_1(U_0),$$

for some $\varepsilon \le \varepsilon_0$ universal. Then there exists a sliding A with $|A - I| \le C\varepsilon$ such that

$$|u(Ax) - U_0| \le C\varepsilon(|x'|^2 + x_n^{2+\alpha})^{1+\frac{\beta}{2}}.$$

Next, we state the $C^{2,\beta}$ estimates in a neighborhood of the origin for solutions to (2.1) which are perturbations of U_0.

Proposition 2.4 ($C^{2,\beta}$ estimates). *Assume u satisfies* (2.1) *and* (2.3) *with*

$$|u - U_0| \le \varepsilon_0, \quad [g]_{C^\gamma} \le \varepsilon_0, \quad [D^2\varphi]_{C^\beta} \le \varepsilon_0, \quad \|q\|_{C^{2,\beta}} \le \varepsilon_0,$$

and $\beta \in (0, \frac{2}{2+\alpha})$, $\beta \le \alpha$, $\gamma = \frac{2+\alpha}{2}\beta$, and ε_0 small depending on n, β, α. Then

$$\|u\|_{C^{2,\beta}(\overline{\Omega} \cap S_{1/2}(U_0))} \le C.$$

As mentioned above, to reduce the proof of Theorem 2.2 (or Theorem 2.1) to that of Proposition 2.4 (or Proposition 2.3), we need to show that by suitably rescaling u and the domain Ω, the new function is well approximated by U_0 on a new domain with very flat boundary near the origin. More precisely, we have the following rescaling lemma whose proof we postpone till Section 6. We recall that we denote by F_h the diagonal matrix

$$F_h = \text{diag}(h^{\frac{1}{2}}, h^{\frac{1}{2}}, .., h^{\frac{1}{2}}, h^{\frac{1}{2+\alpha}}).$$

Lemma 2.5 (Rescaling). *Suppose that u satisfies the hypotheses of Theorem 2.2. Then there exist two linear transformations D symmetric, and A a sliding, with $|D|, |A| \leq C$ such that the rescaling (u_h, Ω_h) of (u, Ω) given by*

$$u_h(x) = \frac{u(DAF_h x)}{h}, \qquad x \in \Omega_h = (DAF_h)^{-1}\Omega,$$

and the boundary data φ_h, q_h of (u_h, Ω_h) satisfy in $S_1(U_0) \cap \Omega_h$ the hypothesis of Proposition 2.4, provided that $h = h(\varepsilon_0)$ is chosen small depending on the universal constants and ε_0.

We now say a few words about the proofs of the above propositions.

We prove Propositions 2.4 by combining the interior $C^{2,\beta}$ estimates together with pointwise $C^{2,\beta}$ estimates in Proposition 2.3 applied at points on the boundary. The geometric properties of sections of u near the boundary play a crucial role.

The key ingredient in the proof of Proposition 2.3 is the following iteration lemma.

Lemma 2.6 (Iteration step). *Assume that for some small $\varepsilon \leq \varepsilon_0$, u satisfies (2.2), (2.3),*

$$|u - U_0| \leq \varepsilon \qquad in \quad \Omega \cap S_1(U_0),$$

and in this set

$$(1 + \delta_0 \varepsilon)x_n^\alpha \geq \det D^2 u \geq (1 - \delta_0 \varepsilon)\left((x_n - \delta_0 \varepsilon)^+\right)^\alpha.$$

If

$$|x_n| \leq \delta_0 \varepsilon |x'|^2 \quad and \quad \left|\varphi - \frac{1}{2}|x'|^2\right| \leq \delta_0 \varepsilon |x'|^2 \qquad on \quad \partial\Omega \cap S_1(U_0),$$

then there exists a rescaling $\tilde{u}$ of u at height $h = \theta_0$,

$$\tilde{u}(x) := \frac{1}{h}u(A_h F_h x), \quad x \in \tilde{\Omega} := (A_h F_h)^{-1}\Omega \quad with \quad h = \theta_0, \quad |A_h - I| \leq C\varepsilon,$$

such that

$$|\tilde{u} - U_0| \leq \varepsilon \theta_0^{\frac{\beta}{2}} \qquad in \quad \tilde{\Omega} \cap S_1(U_0).$$

The constants above ε_0, δ_0, θ_0 (small) and C are universal: they depend only on n, α and β.

The proof of Lemma 2.6 follows the standard perturbation method: we approximate $u - U_0$ by solutions of the Grushin-type operator

$$L w = 0, \qquad L w := x_n^\alpha \Delta_{x'} w + w_{nn},$$

which is the linearized equation of the Monge-Ampère equation

$$\det D^2 U_0 = x_n^\alpha.$$

Before we proceed with the proof we state the $C^{2, \frac{2}{2+\alpha}}$ estimates for solutions of such linearized equation.

Lemma 2.7. *Assume that for some $\alpha > 0$, w solves*

(2.4) $$L w := x_n^\alpha \Delta_{x'} w + w_{nn} = 0 \ in \ B_1^+, \ w = 0 \ on \ \{x_n = 0\}$$

and $\|w\|_{L^\infty(B_1^+)} \leq 1$. Then in $B_{1/2}^+$, w satisfies

$$\left|w(x) - \left(a_0 + a' \cdot x'\right)x_n\right| \leq C_0 \left(|x'|^2 + x_n^{2+\alpha}\right)^{\frac{3+\alpha}{2+\alpha}}$$

with C_0 universal, and $a_0 \in \mathbb{R}, a' \in \mathbb{R}^{n-1}$ with $|a_0|$, $|a'|$ bounded by a universal constant (i.e. depending only on n and α).

The proof of Lemma 2.7 is standard and will be given in Section 6.

3. Proof of Proposition 2.3

In this section, we prove Proposition 2.3. We start with the proof of the main lemma.

Proof of Lemma 2.6. Assume u satisfies the hypotheses of the lemma and denote

$$v := \frac{u - U_0}{\varepsilon}.$$

Step 1: We show that v is well approximated by solutions to the linearized equation (2.4). Precisely, for any small $\eta > 0$ we can find a solution w to

$$L\,w = 0, \ w = 0 \text{ on } \{x_n = 0\}$$

such that

$$|v - w| \le \eta \qquad \text{in} \quad S_{1/2}(U_0) \cap \overline{\Omega},$$

provided that $\varepsilon_0(\eta)$, $\delta_0(\eta)$ are chosen sufficiently small depending on η and the universal constants.

In other words, we need to show that for a sequence of ε_0, $\delta_0 \to 0$ the corresponding v's converge uniformly in $S_{1/2}(U_0) \cap \overline{\Omega}$ to a solution of $Lw = 0$ (along a subsequence).

First we show that v separates almost "linearly" away from $\{x_n = 0\}$.

For each $x_0 \in S_{1/2}(U_0) \cap \{x_n = 0\}$ we consider the upper barrier for u:

$$\bar{u}_{x_0}(x) = \frac{1}{2}|x'|^2 + 8\varepsilon|x' - x_0'|^2 + \frac{1 - \delta_0\varepsilon}{(1 + 16\varepsilon)^{n-1}} \frac{((x_n - \delta_0\varepsilon)^+)^{2+\alpha}}{(1 + \alpha)(2 + \alpha)} + C\varepsilon x_n + 2\delta_0\varepsilon,$$

for some C large to be determined.

Then, we have

(3.1) $$\det D^2 \bar{u}_{x_0} = (1 - \delta_0\varepsilon)\left((x_n - \delta_0\varepsilon)^+\right)^\alpha \le \det D^2 u \text{ in } S_1(U_0) \cap \Omega,$$

and we easily find that on $\partial S_1(U_0) \cap \Omega$,

$$\bar{u}_{x_0} \ge U_0 + 8\varepsilon|x' - x_0'|^2 + \frac{C}{2}\varepsilon x_n \ge 1 + \varepsilon \ge u,$$

and on $\partial\Omega \cap S_1(U_0)$,

$$\bar{u}_{x_0} \ge \frac{1}{2}|x'|^2 + 2\delta_0\varepsilon \ge u.$$

By the maximum principle we obtain $u \le \bar{u}_{x_0}$ in $\Omega \cap S_1(U_0)$ which implies that

$$v(x_0', x_n) \le 2\delta_0 + Cx_n.$$

The opposite inequality follows similarly by considering the lower barrier

$$\underline{u}_{x_0}(x) = \frac{1}{2}|x'|^2 - 8\varepsilon|x' - x_0'|^2 + \frac{1 + \delta_0\varepsilon}{(1 - 16\varepsilon)^{n-1}} \frac{x_n^{2+\alpha}}{(1 + \alpha)(2 + \alpha)} - C\varepsilon x_n - 2\delta_0\varepsilon.$$

Then

$$\det D^2 \underline{u} = (1 + \delta_0\varepsilon)x_n^\alpha \ge \det D^2 u \text{ in } S_1(U_0) \cap \Omega,$$

and on $\partial S_1(U_0) \cap \Omega$,

$$\underline{u}_{x_0} \le U_0 - 8\varepsilon|x' - x_0'|^2 - \frac{C}{2}\varepsilon x_n \le 1 - \varepsilon \le u,$$

and on $\partial\Omega \cap S_1(U_0)$,

$$\underline{u}_{x_0} \le \frac{1}{2}|x'|^2 - 2\delta_0\varepsilon \le u.$$

By the maximum principle we obtain $\underline{u}_{x_0} \le u$ in $\Omega \cap S_1(U_0)$ which implies that

$$v(x_0', x_n) \ge -2\delta_0 - Cx_n.$$

In conclusion

(3.2) $-2\delta_0 - Cx_n \leq v \leq 2\delta_0 + Cx_n \quad \text{in} \quad S_{1/2}(U_0) \cap \overline{\Omega}.$

This inequality provides the uniform convergence of the v's near $\{x_n = 0\}$. It remains to prove the uniform convergence on compact subsets of $S_1(U_0) \cap \{x_n > 0\}$.

Fix a ball $B_{2\rho} := B_{2\rho}(z_0) \subset S_1(U_0)$ and consider the solution u_0 to

$$\det D^2 u_0 = x_n^\alpha \quad \text{in} \quad B_{\frac{3}{2}\rho}, \quad u_0 = u \quad \text{on} \quad \partial B_{\frac{3}{2}\rho},$$

and the corresponding

$$v_0 := \frac{u_0 - U_0}{\varepsilon}.$$

We claim that if $\rho \geq 2\varepsilon\delta_0$ then

(3.3) $|u - u_0| \leq C(\rho)\varepsilon\delta_0 \quad \text{in} \quad B_{\frac{3}{2}\rho}(z_0).$

Indeed, notice that in this ball $B_{\frac{3}{2}\rho}(z_0)$, we have

$$|\det D^2 u - \det D^2 u_0| \leq C(\rho)\varepsilon\delta_0, \quad \text{and} \quad \det D^2 u, \ \det D^2 u_0 \geq c(\rho).$$

Now we can use maximum principle and the matrix inequality

$$\det(A + \lambda I) \geq \det A + n\lambda(\det A)^{\frac{n-1}{n}}, \quad \text{if} \quad A \geq 0, \lambda \geq 0,$$

and obtain in $B_{\frac{3}{2}\rho}(z_0)$

$$u + \Psi \leq u_0, \quad u_0 + \Psi \leq u, \quad \text{where} \quad \Psi := C'(\rho)\,\delta_0\varepsilon\left(|x - z_0|^2 - (\frac{3}{2}\rho)^2\right),$$

for some large $C'(\rho)$ and this proves our claim (3.3).

From (3.3) we find

$$|v - v_0| \leq C(\rho)\delta_0 \quad \text{in} \quad B_{\frac{3}{2}\rho}(z_0),$$

hence $v - v_0 \to 0$ uniformly in $B_{\frac{3}{2}\rho}(z_0)$ as $\delta_0 \to 0$.

Next we show that, as $\varepsilon_0 \to 0$, the corresponding v_0's converge uniformly (along a subsequence) in $\overline{B_\rho}(z_0)$ to a solution of $Lw = 0$. Note that

$$0 = \frac{1}{\varepsilon}(\det D^2 u_0 - \det D^2 U_0) = Tr(A\,D^2 v_0)$$

where

$$A := \int_0^1 \text{cof}(D^2 U_0 + t(D^2 u_0 - D^2 U_0))dt.$$

Here $\text{cof}\,M$ represents the cofactor matrix of M.

As $\varepsilon_0 \to 0$ we have $\varepsilon \to 0$ and therefore $D^2 u_0 \to D^2 U_0$ uniformly in $\overline{B_\rho}(z_0)$. This gives $A \to \text{cof}\,D^2 U_0$ uniformly in $\overline{B_\rho}(z_0)$, thus v_0 must converge to a solution of the linearized equation for U_0.

Step 2: We apply Lemma 2.7 for the function w in Step 1 and obtain

(3.4) $\left|v(x) - (a_0 + a' \cdot x')x_n\right| \leq \eta + C_0\left(|x'|^2 + x_n^{2+\alpha}\right)^{\frac{3+\alpha}{2+\alpha}}$

for some $a_0 \in \mathbb{R}$, $a' = (a_1, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$, with $|a_0|, |a'|$ bounded by a constant depending on n and α.

We choose $\eta = C_0\theta_0^{\frac{3+\alpha}{2+\alpha}}$ for some small θ_0 to be specified later, and we find

(3.5) $\left|u - \left(\varepsilon a_0 x_n + U_0(x' + \varepsilon a' x_n, x_n)\right)\right| \leq \left(2C_0\theta_0^{\frac{3+\alpha}{2+\alpha}} + C\varepsilon\right)\varepsilon \leq 3C_0\theta_0^{\frac{3+\alpha}{2+\alpha}}\varepsilon \quad \text{in} \quad S_{\theta_0}(U_0).$

Step 3: Next we show that, $\nabla u(0) = 0$ implies

(3.6)
$$|a_0| \leq C_1 \theta_0,$$

for some C_1 universal, depending only on n and α.

If, by contradiction, $a_0 > C_1 \theta_0$ then we construct the lower barrier

$$\underline{u} := U_0(x' + \varepsilon a' x_n, x_n) + \varepsilon \left(-C' \theta_0^{\frac{1}{2+\alpha}} \sum_{i=1}^{n-1} (x_i + \varepsilon a_i x_n)^2 + C'' \theta_0^{\frac{1}{2+\alpha}} x_n^{2+\alpha} + \frac{a_0}{2} x_n \right),$$

with $C' = 4C_0$ and C'' large such that

$$\det D^2 \underline{u} \geq (1 + \theta_0^{\frac{1}{2+\alpha}} \varepsilon) x_n^\alpha \geq (1 + \delta_0 \varepsilon) x_n^\alpha \geq \det D^2 u,$$

provided that δ_0 is sufficiently small. We also have $\underline{u} \leq u$ on $\partial(S_{\theta_0}(U_0) \cap \Omega)$.

Indeed, on $\partial S_{\theta_0}(U_0) \cap \Omega$, in view of (3.5), it follows that

$$\underline{u} \leq U_0(x' + \varepsilon a' x_n, x_n) + \varepsilon a_0 x_n - \varepsilon \left(C' \theta_0^{\frac{1}{2+\alpha}} \sum_{i=1}^{n-1} (x_i + \varepsilon a_i x_n)^2 + \frac{a_0}{4} x_n \right)$$

$$\leq U_0(x' + \varepsilon a' x_n, x_n) + \varepsilon a_0 x_n - \varepsilon \left(4C_0 \theta_0^{\frac{1}{2+\alpha}} |x'|^2 + \frac{C_1}{4} \theta_0 x_n + C\varepsilon \right)$$

$$\leq U_0(x' + \varepsilon a' x_n, x_n) + \varepsilon a_0 x_n - 3C_0 \theta_0^{\frac{3+\alpha}{2+\alpha}} \varepsilon \leq u,$$

and on $\partial \Omega \cap S_{\theta_0}(U_0)$ we use that $x_n \leq \delta_0 \varepsilon |x'|^2$ and $|a_0| \leq C$ and obtain

$$\underline{u} \leq \left(\frac{1}{2} - \frac{C'}{2} \theta_0^{\frac{1}{2+\alpha}} \varepsilon + C\delta_0 \varepsilon \right) |x'|^2 \leq \left(\frac{1}{2} - \delta_0 \varepsilon \right) |x'|^2 \leq u.$$

By the maximum principle we obtain that $\underline{u} \leq u$ in $S_{\theta_0}(U_0) \cap \Omega$ hence $u_n(0) \geq \underline{u}_n(0) > 0$ and we reach a contradiction.

If we assume $a_0 < -C_1 \theta_0$ then we consider the upper barrier

$$\bar{u} := \left(\frac{1}{2} + \varepsilon C' \theta_0^{\frac{1}{2+\alpha}} \right) |x' + \varepsilon a' x_n|^2 + \frac{1 - C'' \varepsilon \theta_0^{\frac{1}{2+\alpha}}}{(1+\alpha)(2+\alpha)} \left((x_n - \delta_0 \varepsilon)^+ \right)^{2+\alpha} + \frac{\varepsilon a_0}{2} x_n.$$

Then we obtain

$$\det D^2 \bar{u} \leq (1 - \delta_0 \varepsilon) \left((x_n - \delta_0 \varepsilon)^+ \right)^\alpha \leq \det D^2 u,$$

and, similarly as above, we obtain $\bar{u} \geq u$ on $\partial(S_{\theta_0}(U_0) \cap \Omega)$. By maximum principle we find $u \leq \bar{u}$ in $S_{\theta_0}(U_0) \cap \Omega$ hence $u_n(0) \leq \bar{u}_n(0) < 0$ and again we reach a contradiction. Thus (3.6) is proved.

Step 4: We use (3.6) in (3.4) and obtain

$$\left| v(x) - a' \cdot x' \, x_n \right| \leq C_2 \theta_0^{\frac{3+\alpha}{2+\alpha}} \quad \text{in} \quad S_{2\theta_0}(U_0).$$

We choose θ_0 small, depending also on $\beta \in (0, \frac{2}{2+\alpha})$, such that

$$C_2 \theta_0^{\frac{3+\alpha}{2+\alpha}} \leq \frac{1}{2} \theta_0^{1+\frac{\beta}{2}},$$

and we easily obtain

$$|u - U_0(x' + \varepsilon a' x_n, x_n)| \leq \left(\frac{1}{2} \theta_0^{1+\frac{\beta}{2}} + C\varepsilon \right) \varepsilon \leq \theta_0^{1+\frac{\beta}{2}} \varepsilon \quad \text{in} \quad S_{2\theta_0}(U_0).$$

Hence if $\varepsilon \leq \varepsilon_0$ is small,

$$|u(Ax) - U_0(x)| \leq \theta_0^{1+\frac{\beta}{2}} \varepsilon \quad \text{in} \quad S_{\theta_0}(U_0)$$

where $Ax := x - \varepsilon(a', 0) x_n$. $\qquad \square$

We are now ready to give the proof of Proposition 2.3.

Proof of Proposition 2.3. It is straightforward to check that, since $|x_n| \leq \varepsilon|x'|^2$ on $\partial\Omega$, the distance function $d_{\partial\Omega}$ satisfies

$$x_n \geq d_{\partial\Omega}(x) \geq \left(x_n - 2\varepsilon|x'|^2\right)^+.$$

We will prove the proposition for more general right hand sides in which we replace $d_{\partial\Omega}$ by the inequality above. Precisely we assume that u satisfies

$$(3.7) \qquad (1 - \varepsilon|x|^\gamma)\left[\left(x_n - 2\varepsilon|x'|^2\right)^+\right]^\alpha \leq \det D^2 u \leq (1 + \varepsilon|x|^\gamma)\, x_n^\alpha.$$

Let $\varepsilon_0, \delta_0, \theta_0$ be as in Lemma 2.6.

We prove by induction that there exists a sequence of slidings A_k such that the rescalings u_k at height $h = \theta_0^k$

$$u_k := \frac{1}{h} u(A_k F_h x), \quad x \in \tilde\Omega := (A_k F_h)^{-1}\Omega, \qquad h = \theta_0^k,$$

satisfy

$$(3.8) \qquad |u_k - U_0| \leq \bar\varepsilon_k := 8\delta_0^{-1}\varepsilon\theta_0^{k\frac{\beta}{2}} \text{ in } S_1(U_0) \cap \tilde\Omega, \qquad |A_k - A_{k-1}| \leq C\bar\varepsilon_{k-1}.$$

Clearly, this holds for $k = 0$ with $A_0 = I$ from the hypotheses of the proposition.

Next we assume it holds for k and prove it for $k + 1$. We check that $\tilde u := u_k$ satisfies the hypotheses of Lemma 2.6 for

$$\tilde\varepsilon := \bar\varepsilon_k \qquad \text{hence} \quad \delta_0\tilde\varepsilon = 8\varepsilon h^{\frac{\beta}{2}}.$$

By (3.8), we have

$$|A_k - I| \leq C\varepsilon.$$

Note that

$$\text{if} \quad \tilde x \in \overline{\tilde\Omega} \cap S_1(U_0), \quad \text{then} \quad \tilde x = F_h^{-1} A_k^{-1} x \quad \text{for some} \quad x \in \overline\Omega,$$

and

$$x_n = h^{\frac{1}{2+\alpha}}\tilde x_n, \qquad |x' - h^{\frac{1}{2}}\tilde x'| \leq C\varepsilon x_n.$$

If $\tilde x \in \partial\tilde\Omega \cap S_1(U_0)$, then $x = A_k F_h \tilde x \in \partial\Omega$, hence $|x_n| \leq \varepsilon|x'|^2$, and we easily find

$$(3.9) \qquad |x' - h^{\frac{1}{2}}\tilde x'| \leq C\varepsilon^2|x'|^2 \leq \varepsilon h|\tilde x'|,$$

and

$$|\tilde x_n| \leq 2\varepsilon h^{\frac{1+\alpha}{2+\alpha}}|\tilde x'|^2 \leq \delta_0\bar\varepsilon_k|\tilde x'|^2.$$

Moreover, the boundary value of $\tilde u$ is

$$\tilde\varphi(\tilde x') = \frac{1}{h}\varphi(x')$$

hence, using (3.9)

$$\left|\tilde\varphi(\tilde x') - \frac{1}{2}|\tilde x'|^2\right| \leq \frac{1}{h}\left|\varphi(x') - \frac{1}{2}|x'|^2\right| + 2\varepsilon h^{\frac{1}{2}}|\tilde x'|^2$$

$$\leq \frac{1}{h}\varepsilon|x'|^{2+\beta} + 2\varepsilon h^{\frac{1}{2}}|\tilde x'|^2$$

$$\leq 4\varepsilon h^{\frac{\beta}{2}}|\tilde x'|^2 \leq \delta_0\bar\varepsilon_k|\tilde x'|^2.$$

If $\tilde x \in \tilde\Omega \cap S_1(U_0)$ then we denote by $x = A_k F_h \tilde x$ and obtain

$$\det D^2\tilde u(\tilde x) = h^{-\frac{\alpha}{2+\alpha}}\det D^2 u(x).$$

We use (3.7) and $|x| \leq 2h^{\frac{1}{2+\alpha}}$ and obtain

$$(1 - 2\varepsilon h^{\frac{\beta}{2}})\left[\left(\tilde x_n - 8\varepsilon h^{\frac{1}{2+\alpha}}\right)^+\right]^\alpha \leq \det D^2\tilde u \leq (1 + 2\varepsilon h^{\frac{\beta}{2}})\,\tilde x_n^\alpha,$$

or

$$(1 - \delta_0 \bar{\varepsilon}_k) \left[(\tilde{x}_n - \delta_0 \bar{\varepsilon}_k)^+ \right]^\alpha \leq \det D^2 \tilde{u} \leq (1 + \delta_0 \bar{\varepsilon}_k) \, \tilde{x}_n^\alpha.$$

In conclusion we can apply Lemma 2.6 for $\tilde{u}$ and obtain a sliding $\tilde{A}_0$, $|\tilde{A}_0 - I| \leq C\bar{\varepsilon}_k$ such that

$$\left| \frac{1}{\theta_0} \tilde{u}(\tilde{A}_0 F_{\theta_0} x) - U_0 \right| \leq \tilde{\varepsilon} \theta_0^{\frac{\beta}{2}} = \bar{\varepsilon}_{k+1} \quad \text{in} \quad S_1(U_0).$$

This means that u_{k+1} satisfies the desired conclusion by choosing A_{k+1} such that

$$A_{k+1} F_{h\theta_0} = A_k F_h \tilde{A}_0 F_{\theta_0},$$

thus

$$A_{k+1} = A_k \, F_h \tilde{A}_0 F_h^{-1}.$$

Since A_{k+1}, A_k, $\tilde{A}_0$ are slidings, we obtain that the following quantities are comparable up to multiplicative constants

(3.10) $$|A_{k+1} - A_k| \sim |F_h(\tilde{A}_0 - I)F_h^{-1}| \sim h^{\frac{1}{2} - \frac{1}{2+\alpha}} |\tilde{A}_0 - I|,$$

and the induction step is verified.

From (3.10) we conclude that A_k converges geometrically to a limiting sliding A_∞, thus

$$|A_\infty - A_k| \leq C h^{\frac{1}{2} - \frac{1}{2+\alpha}} \bar{\varepsilon}_k,$$

hence

$$A_\infty = A_k \, F_h \tilde{G}_k F_h^{-1} \quad \text{for some sliding } \tilde{G}_k \text{ with} \quad |\tilde{G}_k - I| \leq C\bar{\varepsilon}_k.$$

This means that in (3.8) we may replace the sliding A_k by A_∞ and the right hand side $\bar{\varepsilon}_k$ by $C\bar{\varepsilon}_k$, and the proposition is proved. $\qquad \square$

4. Proof of Proposition 2.4

In this section, we prove Proposition 2.4.

Assume that u satisfies the hypothesis of Proposition 2.3 with

$$[g]_{C^\gamma} \leq \varepsilon \quad \text{in} \quad \Omega \cap S_1(U_0),$$

and from the conclusion of the proposition there exists a sliding A, $|A - I| \leq C\varepsilon$ such that $u(Ax)$ is well approximated by U_0:

$$|u(Ax) - U_0(x)| \leq C\varepsilon U_0(x)^{1 + \frac{\beta}{2}}.$$

For each $h < c$ we let $-t$ be the minimum value of $u - h^{\frac{1+\alpha}{2+\alpha}} x_n$ and x_t the point where is achieved. Thus

$$S_t(x_t) = \{ u < h^{\frac{1+\alpha}{2+\alpha}} x_n \}$$

is a section centered at x_t which is tangent to $\partial\Omega$ at 0. It is easy to check that $t \sim h$ and the sections $S_h(u)$ and $S_t(x_t)$ are comparable in size. Below we estimate the modulus of continuity of $D^2 u$ in the interior of $S_t(x_t)$.

Lemma 4.1. *Assume $\beta \leq \alpha$. In the section $S_{t/4}(x_t)$ we have*

$$\|D^2 u\|_{C^\beta} \leq C, \quad \|D^2 u - D^2 u(0) - x_n^\alpha e_n \otimes e_n\|_{L^\infty} \leq C\varepsilon h^{\frac{\beta}{2}}.$$

Proof. Using that $u(Ax)$ is well approximated by $U_0(x)$ we obtain

$$S_{t/2}(x_t) \subset \mathcal{C} := \left\{ |x'| \leq |x_n| \right\}.$$

Since $\partial\Omega \subset \{ |x_n| \leq \varepsilon |x'|^2 \}$ it follows that in the cone $\mathcal{C}$, $\frac{d_{\partial\Omega}}{x_n}$ is a positive function with gradient bounded by $1 + C\varepsilon$, and in fact

$$\left\| \frac{d_{\partial\Omega}}{x_n} - 1 \right\|_{C^{0,1}(\mathcal{C})} \leq C\varepsilon.$$

This means that in the cone $\mathcal{C}$ (and therefore in $S_{t/2}(x_t)$) we can rewrite the equation for u as

$$\det D^2 u = \bar{g}\, x_n^{\alpha}, \qquad \|\bar{g} - 1\|_{C^{\gamma}} \leq C\varepsilon \quad \text{in } \mathcal{C}.$$

We let u_h denote the rescaled function

$$u_h(x) = \frac{1}{h} u(AF_h x),$$

and let y_t such that $x_t = AF_h y_t$. Notice that

$$AF_h S_{t/(2h)}(y_t) = S_{t/2}(x_t), \qquad S_{t/h}(y_t) = \{u_h < y_n\}$$

where $S_t(y)$ denote the sections for u_h.

We have

$$\det D^2 u_h = \bar{g}_h\, x_n^{\alpha} \quad \text{in} \quad S_{t/(2h)}(y_t),$$

with

$$\bar{g}_h(x) = \bar{g}(AF_h x) \qquad \Rightarrow \qquad \|\bar{g}_h - 1\|_{C^{\gamma}} \leq C\varepsilon h^{\frac{\gamma}{2+\alpha}} = C\varepsilon h^{\frac{\beta}{2}}.$$

From Proposition 2.3 we have

$$|u_h - U_0| \leq C\varepsilon h^{\frac{\beta}{2}} \quad \text{in} \quad S_{t/(2h)}(y_t).$$

Notice that $\gamma > \beta$ hence, by the interior $C^{2,\beta}$ estimates for the Monge-Ampère equation, we obtain

$$\|D^2(u_h - U_0)\|_{C^{\beta}} \leq C\varepsilon h^{\frac{\beta}{2}} \quad \text{in} \quad S_{t/(4h)}(y_t).$$

Rescaling back, we write these inequalities in terms of $D^2 u$ and obtain that in $S_{t/4}(x_t)$

$$\|D^2(u - U_0(A^{-1}x))\|_{L^{\infty}} \leq C\varepsilon h^{\frac{\beta}{2}}, \qquad \|D^2(u - U_0(A^{-1}x))\|_{C^{\beta}} \leq C\varepsilon.$$

We can write

$$D^2(U_0(A^{-1}x)) = Q_0 + x_n^{\alpha} e_n \otimes e_n \qquad \text{and} \quad Q_0 = D^2 u(0).$$

Now we use that $\beta \leq \alpha$ hence $x_n^{\alpha} \in C^{\beta}$ and the proof is completed. $\qquad\square$

Next we check that Proposition 2.3 can be applied at the origin also when $u(0)$ and $\nabla u(0)$ do not necessarily vanish provided that $\partial\Omega$ is $C^{2,\beta}$. Thus the proposition can be used at all other boundary points $z \in \partial\Omega \cap S_{1/2}(U_0)$.

Lemma 4.2. *Assume that u solves*

$$\det D^2 u = g\, d_{\partial\Omega}^{\alpha} \quad in \quad \Omega \cap S_1(U_0),$$

and

$$|u - U_0| \leq \varepsilon, \quad [g]_{C^{\gamma}} \leq \varepsilon, \quad u(x) = \varphi(x') \ on \ \partial\Omega, \quad [D^2\varphi]_{C^{\beta}} \leq \varepsilon,$$
$$\partial\Omega = \{x_n = q(x')\} \quad with \quad \|q\|_{C^{2,\beta}} \leq \varepsilon.$$

Then there exist 2 linear transformations, A sliding, D symmetric, which are $C\varepsilon$ perturbations of the identity I such that

$$|(u - l_0)(DAx) - U_0(x)| \leq C\varepsilon U_0^{1+\frac{\beta}{2}},$$

where $l_0(x) = u(0) + \nabla u(0) \cdot x$ represents the linear part of u at 0.

Proof. We sketch the proof below.

Since $|u - U_0| \leq \varepsilon$ we easily find

$$|g(0) - 1| \leq C\varepsilon, \quad |D_{x'}^2\varphi(0) - I'| \leq C\varepsilon, \quad |\nabla_{x'}\varphi(0)| \leq C\varepsilon.$$

Using standard barriers at the origin we find $|u_n(0)| \leq C\varepsilon$ hence the linear part l_0 of u at the origin satisfies

$$l_0(x) = u(0) + \nabla_{x'}\varphi(0)x' + u_n(0)x_n = O(\varepsilon).$$

The function $w = u - l_0$ satisfies the same equation as u and $w(0) = 0$, $\nabla w(0) = 0$ and its boundary values on $\partial\Omega$ equal

$$\varphi_w(x') = \varphi(x') - l_0(x', q(x')) \quad \text{hence} \quad \|D^2_{x'}\varphi_w - I'\|_{C^{2,\beta}} \le C\varepsilon.$$

Now we may choose a symmetric matrix D which has e_n as an eigenvector such that

$$\tilde{w}(x) = w(Dx), \quad x \in \tilde{\Omega} := D^{-1}\Omega$$

satisfies the hypotheses of Proposition 2.3 for its boundary data $\varphi_{\tilde{w}}$ and right hand side $g_{\tilde{w}}d^\alpha_{\partial\tilde{\Omega}}$ with

$$D^2_{x'}\varphi_{\tilde{w}}(0) = I', \qquad g_{\tilde{w}}(0) = 1.$$

In conclusion there exists a sliding A which is a $C\varepsilon$ perturbation of the identity I such that $|\tilde{w}(Ax) - U_0(x)| \le C\varepsilon U_0(x)^{1+\frac{\beta}{2}}$. $\qquad\square$

We write $U_0(A^{-1}D^{-1}x)$ as a quadratic polynomial plus a multiple of $x^{2+\alpha}$.
Since $U_0(A^{-1}D^{-1}x) \le C|x|^2$, the conclusion in the lemma above gives

$$(4.1) \qquad \left| u - \left(l_0 + \frac{1}{2}x^T D^2 u(0)x + \frac{a(0)}{(1+\alpha)(2+\alpha)}x_n^{2+\alpha} \right) \right| \le C\varepsilon|x|^{2+\beta}.$$

The coefficient $a(0)$ and the tangential determinant of $D^2_{x'}u$ satisfy

$$(4.2) \qquad a(0)\det D^2_{x'}u(0) = g(0).$$

Clearly the conclusion holds for all points $z \in \partial\Omega \cap S_{1/2}(U_0)$ i.e.

$$\left| u - \left(l_z + \frac{1}{2}(x-z)^T D^2 u(z)(x-z) + \frac{a(z)}{(1+\alpha)(2+\alpha)}((x-z)\cdot\nu_z)^{2+\alpha} \right) \right| \le C\varepsilon|x-z|^{2+\beta}.$$

Next we show that $D^2 u$ and a are C^β on $\partial\Omega \cap S_{1/2}(U_0)$.

Lemma 4.3. On $\partial\Omega \cap S_{1/2}(U_0)$, we have

$$[D^2 u]_{C^\beta} \le C\varepsilon, \qquad [a]_{C^\beta} \le C\varepsilon.$$

Proof. Since

$$d_{\partial\Omega} \le x_n \le d_{\partial\Omega} + C\varepsilon|x|^2,$$

it follows that

$$|x_n^{2+\alpha} - d_{\partial\Omega}^{2+\alpha}| \le C\varepsilon|x|^2 d^{1+\alpha} + C(\varepsilon|x|^2)^{2+\alpha} \le C\varepsilon|x|^{3+\alpha}.$$

Also, from the previous lemma we have $|a(0) - 1| \le C\varepsilon$. Using these in (4.1) together with $\beta \le \alpha$ and we find

$$\left| u - \left(l_0 + \frac{1}{2}x^T D^2 u(0)x + \frac{1}{(1+\alpha)(2+\alpha)}d_{\partial\Omega}^{2+\alpha} \right) \right| \le C\varepsilon|x|^{2+\beta}.$$

Subtracting the corresponding inequality at z we find

$$\left| l_0 - l_z + \frac{1}{2}x^T D^2 u(0)x - \frac{1}{2}(x-z)^T D^2 u(z)(x-z) \right| \le C\varepsilon|z|^{2+\beta} \qquad \forall x \in B_{|z|} \cap \Omega$$

which gives

$$|D^2 u(0) - D^2 u(z)| \le C\varepsilon|z|^\beta.$$

Now the corresponding inequality for $a(z)$ follows easily from the equality (4.2) and the fact that $[g]_{C^\beta} \le C\varepsilon$. $\qquad\square$

We now give the proof of Proposition 2.4.

Proof of Proposition 2.4. Let x_0 and x_1 be two points in $\Omega \cap S_{1/4}(U_0)$ and let $S_{t_0}(x_0)$, $S_{t_1}(x_1)$ be the the maximal sections included in Ω which become tangent to $\partial\Omega$ at, say 0 for simplicity, and respectively z. Let $d_0 := |x_0|$, $d_1 := |x_1 - z|$ and assume $d_0 \geq d_1$. Next we show that

$$|D^2 u(x_1) - D^2(x_0)| \leq C|x_1 - x_0|^\beta.$$

We distinguish 3 cases depending on the positions of x_0, x_1 and z.

Case 1: $x_1 \in S_{t_0/4}(x_0)$. Then we're done by Lemma 4.1.

Case 2: $x_1 \notin S_{t_0/4}(x_0)$ and $|z|^2 \leq c\, d_0^{2+\alpha}$.

Then $|x_0 - x_1| \geq cd_0$. This inequality follows by inspecting the corresponding sections for the rescaling u_h with $h = d_0^{2+\alpha}$ and noticing that their centers y_0 and y_1 must satisfy $(y_0 - y_1) \cdot e_n \geq c|y_0|$. We rescale back and obtain $|x_0 - x_1| \geq cd_0$.

Moreover by Lemma 4.1 and $\alpha \geq \beta$ we have

$$|D^2 u(x_0) - D^2 u(0)| \leq C(x_0 \cdot e_n)^\alpha + C\varepsilon d_0^{(2+\alpha)\frac{\beta}{2}} \leq Cd_0^\beta,$$

and similarly

$$|D^2 u(x_1) - D^2 u(z)| \leq Cd_1^\beta.$$

Now we obtain the desired inequality since by Lemma 4.3, $|D^2 u(z) - D^2 u(0)| \leq C\varepsilon z^\beta \leq Cd_0^\beta$.

Case 3: $|z|^2 \geq c\, d_0^{2+\alpha}$.

Then $|x_0 - x_1| \geq c|z|$. This follows by inspecting the images of the sections under the rescaling u_h with $h = |z|^2$ and noticing that their centers y_0 and y_1 must satisfy $|y_0 - y_1| \geq c$.

We use Lemma 4.1 at 0 and z and obtain

$$D^2 u(x_0) = D^2 u(0) + a(0)\,(x_0 \cdot e_n)^\alpha\, e_n \otimes e_n + O(\varepsilon |z|^\beta),$$

$$D^2 u(x_1) = D^2 u(z) + a(z)\,((x_1 - z) \cdot \nu_z)^\alpha\, \nu_z \otimes \nu_z + O(\varepsilon |z|^\beta).$$

Now the the desired result is a consequence of Lemma 4.3, $|x_0 - x_1| \geq c|z|$, and

$$|e_n - \nu_z| \leq C\varepsilon|z|,$$

$$|x_0 \cdot e_n - (x_1 - z) \cdot \nu_z| \leq C|x_0 - x_1| + C|z| \leq C|x_0 - x_1|.$$

$\square$

5. GLOBAL C^∞ REGULARITY OF MONGE-AMPÈRE EIGENFUNCTIONS

In this section, we prove global C^∞ regularity of Monge-Ampère eigenfunctions stated in Theorem 1.4. For convenience, we restate it here.

Theorem 5.1. *Let Ω be a bounded and uniformly convex domain in $\mathbb{R}^n$. Assume $\partial\Omega \in C^\infty$ and u satisfies*

$$(5.1) \qquad \begin{cases} (\det D^2 u)^{\frac{1}{n}} = \lambda|u| & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \in C^\infty(\overline{\Omega})$.

To prove the above theorem, we perform first a Hodograph transform and reduce (5.1) to a similar equation in the upper half-space. Then we perform the partial Legendre transform in the nondegenerate x' coordinates. The structure of the equation satisfied by the transformed function allows us to use the $C^{2,\beta}$ estimates for the Monge-Ampère eigenfunctions obtained in Theorem 1.3 together with Schauder estimates for linear equations with Hölder coefficients modeled by a degenerate Grushin-type operator as in (2.4). These give the desired global C^∞ regularity.

5.1. **Equivalent equation in the upper half-space.** We first write an equation in the upper half-space that is locally equivalent to (5.1). After a dilation we may assume that $\lambda = 1$.

The Monge-Ampère eigenfunctions are C^∞ in the interior of Ω so it remains to prove their C^∞ smoothness near the boundary $\partial\Omega$. Assume that $0 \in \partial\Omega$ and e_n is the inner normal of $\partial\Omega$ at 0. We make the rotation of coordinates

$$y_n = -x_{n+1}, \ y_{n+1} = x_n, \ y_k = x_k \ (1 \le k \le n-1).$$

In the new coordinates, the graph of u near the origin can be represented as $y_{n+1} = \tilde{u}(y)$ in the upper half-space $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_n > 0\}$. The Gauss curvature is

$$K = \frac{\det D^2 u}{(1 + |Du|^2)^{\frac{n+2}{2}}} = \frac{|u|^n}{(1 + |Du|^2)^{\frac{n+2}{2}}}.$$

The tangent plane at 0 is given by

$$x_{n+1} - u_n x_n - u_k x_k = 0.$$

After the above rotation of coordinates, it is given by

$$-y_n - u_n y_{n+1} - u_k y_k = 0, \ \text{or} \ y_{n+1} + \frac{y_n}{u_n} + \frac{u_k}{u_n} y_k = 0.$$

Hence

$$\tilde{u}_{y_n} = -\frac{1}{u_{x_n}} > 0 \ (\text{near } 0), \ \tilde{u}_{y_k} = -\frac{u_k}{u_n}.$$

Now, from (5.1) we obtain the following equation in a neighborhood of the origin in the upper half-space $\{y_n > 0\}$:

$$\det D^2 \tilde{u} = K(1 + |D\tilde{u}|^2)^{\frac{n+2}{2}} = |y_n|^n \left(\frac{1 + |D\tilde{u}|^2}{1 + |Du|^2}\right)^{\frac{n+2}{2}} = |y_n|^n (\frac{1}{u_n^2})^{\frac{n+2}{2}} = y_n^n \ \tilde{u}_n^{n+2}.$$

Near the origin, the boundary $\partial\Omega$ is given by $x_n = \phi(x')$ in the original coordinates. Thus, the boundary condition for $\tilde{u}$ is $\tilde{u} = \phi$ on $\{y_n = 0\}$. Thus, locally, we have for some small $r_0 > 0$ (now relabeling y by x)

(5.2)
$$\begin{cases} \det D^2 \tilde{u} &= x_n^n \ \tilde{u}_n^{n+2} \quad \text{in } B_{r_0}^+, \\ \tilde{u} &= \phi \qquad\quad \text{on } \{x_n = 0\} \cap B_{r_0}. \end{cases}$$

We remark that since $u \in C^{2,\beta}(\overline{\Omega})$ by Theorem 1.3, we have $\tilde{u} \in C^{2,\beta}(\overline{B}_{r_0}^+)$ for some small $\beta > 0$, and $\tilde{u}_n > c$. It remains to show that solutions of (5.2) with $\phi \in C^\infty$ are smooth up to the boundary in a neighborhood of the origin. For simplicity of notation, we relabel $\tilde{u}$ from (5.2) by u.

5.2. **Partial Legendre transform.** Next we perform the following partial Legendre transformation to the solutions of (5.2):

(5.3)
$$y_i = u_i(x) \quad (i \le n-1), \quad y_n = x_n, \qquad u^*(y) = x' \cdot \nabla_{x'} u - u(x).$$

The function u^* is obtained by taking the Legendre transform of u on each slice $x_n = $ constant. Note that

$$(u^*)^* = u.$$

We claim that if u satisfies (5.2) then u^* (which is convex in y' and concave in y_n) satisfies

(5.4)
$$\begin{cases} y_n^\alpha(-u_n^*)^{n+2} \det D_{y'}^2 u^* + u_{nn}^* &= 0 \text{ in } B_\delta^+ \\ u^* &= \phi^* \text{ on } \{y_n = 0\} \cap B_\delta, \end{cases}$$

where $\alpha = n$. Moreover $u^* \in C^{2,\beta}(\overline{B}_\delta^+)$, $-u_n^* > c$ and $\phi^* \in C^\infty$.

There are several ways to see this. A direct way is to compute the transformation matrices $Y = (Y_{ij}) = \left(\frac{\partial y_i}{\partial x_j}\right)$ and $Y^{-1} = X = (X_{ij}) = \left(\frac{\partial x_i}{\partial y_j}\right)$. We obtain

$$Y_{ij} = \left\{ \begin{array}{ll} u_{ij} & \text{if } i \leq n-1 \\ \delta_{jn} & \text{if } i = n \end{array} \right. , \quad X = \left[\begin{array}{c|c} (D^2_{x'}u)^{-1} & U^{ni}/\det D^2_{x'}u \\ \hline 0 & 1 \end{array} \right]$$

where (U^{ij}) denotes the cofactor matrix of D^2u.

Using these transformations, we find that

$$u^*_j = x_j \text{ if } j < n, \ u^*_n = -u_n$$

and therefore

$$D^2_{y'}u^* = (D^2_{x'}u)^{-1}, \ u^*_{nn} = -\sum_{k \leq n} \frac{\partial u_n}{\partial x_k}\frac{\partial x_k}{\partial y_n} = -\sum_{k < n} u_{nk}\frac{U^{nk}}{\det D^2_{x'}u} - u_{nn}.$$

Since $\det D^2_{y'}u^* = (\det D^2_{x'}u)^{-1} = 1/U^{nn}$, it follows that

$$\frac{-u^*_{nn}}{\det D^2_{y'}u^*} = \left(\sum_{k<n} u_{nk}\frac{U^{nk}}{\det D^2_{x'}u} + u_{nn} \right)\det D^2_{x'}u = \sum_{k \leq n} u_{nk}U^{nk} = \det D^2u.$$

Hence u^* satisfies (5.4). It remains to prove that u^* is smooth up to the boundary in a neighborhood of the origin.

5.3. Schauder estimates for linear equations.
In order to obtain the smoothness of u^* from (5.4) we establish Schauder estimates for its linearized equation. We consider linear equations of the form

$$(5.5) \qquad\qquad x_n^\alpha \sum_{i,j \leq n-1} a^{ij}v_{ij} + v_{nn} = x_n^\alpha f(x)$$

with a^{ij} uniformly elliptic

$$\lambda I' \leq (a^{ij}(x))_{i,j} \leq \Lambda I'.$$

Definition 5.2. *We define the distance d_α between 2 points y and z in the upper half-space by*

$$d_\alpha(y,z) := |y' - z'| + \left| y_n^{\frac{2+\alpha}{2}} - z_n^{\frac{2+\alpha}{2}} \right|.$$

Notice that d_α is equivalent to the distance induced by the metric

$$ds^2 = dx'^2 + x_n^\alpha dx_n^2.$$

The relation between d_α and the Euclidean distance in $\overline{B}_1^+$ is as follows:

$$(5.6) \qquad\qquad c|y-z|^{\frac{2+\alpha}{2}} \leq d_\alpha(y,z) \leq C|y-z|,$$

$$d_\alpha(y,z) \sim |y-z| \quad \text{if} \quad y,z \in \overline{B}_1^+ \cap \{x_n \geq 1/4\}.$$

The distance d_α occurs naturally in equation (5.5) since they both scale homogeneously after the transformations $x \mapsto F_h x$ (see Section 6).

If function w is C^γ with respect to d_α (with $\gamma \in (0, \frac{2}{2+\alpha})$) we write

$$w \in C^\gamma_\alpha(\overline{B}_1^+),$$

and define

$$[w]_{C^\gamma_\alpha(\overline{B}_1^+)} = \sup_{y,z \in \overline{B}_1^+, \ y \neq z} \frac{|w(y) - w(z)|}{(d_\alpha(y,z))^\gamma}, \quad \|w\|_{C^\gamma_\alpha(\overline{B}_1^+)} = \|w\|_{L^\infty(\overline{B}_1^+)} + [w]_{C^\gamma_\alpha(\overline{B}_1^+)}.$$

In view of (5.6) we obtain the following relations between the C_α^γ spaces and the standard C^γ Holder spaces :

$$w \in C_\alpha^\gamma(\overline{B}_1^+) \quad \Rightarrow \quad w \in C^\gamma(\overline{B}_1^+)$$

$$w \in C^\beta(\overline{B}_1^+) \quad \Rightarrow \quad w \in C_\alpha^\gamma(\overline{B}_1^+) \quad \text{with} \quad \gamma = \beta \frac{2}{2+\alpha}.$$

Proposition 5.3 (Schauder estimates). *Assume that v solves (5.5) in $\overline{B}_\delta^+$ and*

$$v = \varphi(x') \text{ on } \{x_n = 0\} \cap \overline{B}_\delta^+.$$

If a^{ij}, $f \in C_\alpha^\gamma(\overline{B}_\delta^+)$ with $\frac{\gamma}{2} \leq \frac{\min\{1,\alpha\}}{2+\alpha}$, and $\varphi \in C^{2,\gamma}$, then

$$Dv, D^2v \in C_\alpha^\gamma(\overline{B}_{\delta/2}^+).$$

The proof of this proposition is standard and we postpone it till the last section.

By repeatedly differentiating (5.5) in the x' direction we easily obtain Schauder estimates for higher derivatives. Below $m = (m_1, .., m_{n-1})$ denotes a multi-index with m_i nonnegative integers.

Corollary 5.4. *If in the proposition above $\varphi \in C^{k+2,\gamma}$ for some integer $k \geq 0$ and*

$$D_{x'}^m a^{ij}, D_{x'}^m f \in C_\alpha^\gamma(\overline{B}_\delta^+) \quad \forall m \quad \text{with} \quad |m| \leq k,$$

then

$$DD_{x'}^m v, D^2 D_{x'}^m v \in C_\alpha^\gamma(\overline{B}_{\delta/2}^+) \qquad \forall m \quad \text{with} \quad |m| \leq k.$$

5.4. Proof of Theorem 5.1.
As mentioned at the end of Section 5.2, it suffices to prove C^∞ regularity in a neighborhood of the origin for the function u^* satisfying (5.4):

$$\begin{cases} y_n^\alpha(-u_n^*)^{n+2} \det D_{y'}^2 u^* + u_{nn}^* &= 0 \quad \text{in } B_\delta^+, \\ u^* &= \phi^* \quad \text{on } \{y_n = 0\} \cap B_\delta, \end{cases}$$

with $\alpha = n$, $u^* \in C^{2,\beta}(\overline{B}_\delta^+)$, $-u_n^* > c$ and $\phi^* \in C^\infty$.

Fix $k < n$. Then $v = u_k^*$ solves the linearized equation

(5.7) $$y_n^\alpha \sum_{i,j \leq n-1} a^{ij} v_{ij} + v_{nn} = y_n^\alpha f(y) \quad \text{in} \quad B_\delta^+$$

where

$$a^{ij} = (-u_n^*)^{n+2} U_{y'}^{*ij}, \qquad f(y) = (n+2)(-u_n^*)^{n+1} u_{nk}^* \det D_{y'}^2 u^*.$$

and $U_{y'}^*$ denotes the cofactor matrix of $D_{y'}^2 u^*$.

Since $u^* \in C^{2,\beta}(\overline{B}_\delta^+)$ we obtain Du^*, $D^2 u^* \in C_\alpha^\gamma(\overline{B}_\delta^+)$ for some small $\gamma > 0$, hence a^{ij}, $f \in C_\alpha^\gamma(\overline{B}_\delta^+)$.

By Proposition 5.3, $D^2 v \in C_\alpha^\gamma$ up to the boundary in $\overline{B}_{\delta/2}^+$ which in turn implies $D_{y'} a^{ij}$, $D_{y'} f \in C_\alpha^\gamma(\overline{B}_{\delta/2}^+)$. Now we may apply Corollary 5.4 and iterate this argument to obtain that $D_{y'}^m D_{y_n}^l u^*$ with $l \in \{0, 1, 2\}$ are continuous up to the boundary in $\overline{B}_{\delta/2}^+$ for all multi-indices $m \geq 0$. In order to obtain the continuity of these derivatives for all values of l we differentiate the equation for u^* and use that $\alpha = n$ is a nonnegative integer. Then each derivative $D_{y'}^m D_{y_n}^l u^*$ with $l \geq 3$ can be expressed as a polynomial involving powers of y_n and derivatives $D_{y'}^q D_{y_n}^s u^*$ with $s < l$, thus $u^* \in C^\infty(\overline{B}_{\delta/2}^+)$ as desired. $\square$

6. Rescaling and the linearized equation

In this section, we prove some technical results (Lemmas 2.5 and 2.7 and Proposition 5.3) which were used in the proofs of our main results in the previous sections.

6.1. **Rescaling.** We give below the proof of Lemma 2.5.

Suppose that u satisfies the hypotheses of Theorem 2.2. Assume furthermore that the boundary $\partial\Omega$ is locally given by the graph of a $C^{1,1}$ convex function $q(x')$, i.e.

$$\partial\Omega \cap B_\rho = \{x_n = q(x')\}, \qquad q(0) = 0, \quad \nabla_{x'}q(0) = 0, \quad \|q\|_{C^{1,1}} \le \frac{1}{\rho'}.$$

As mentioned in the Introduction, Lemma 2.5 is a direct consequence of the pointwise C^2 estimate form [S2]. We only need to check that, after an initial rescaling, the equation remains of the same form. First we need the following lemma which compares distance functions during the rescaling process.

Lemma 6.1. *Let A be a sliding along $x_n = 0$ with $|A| \le C$. Let $\Omega_h = (AF_h)^{-1}\Omega$. If $h \le c$ then*

(i) $\partial\Omega_h \cap S_1(U_0)$ is a graph in the e_n direction whose $C^{1,1}$ norm is bounded by $Ch^{\frac{1+\alpha}{2+\alpha}}$.
(ii) the ratio of the distance functions to $\partial\Omega$ and $\partial\Omega_h$ is Lipschitz and

$$\left[\frac{d_{\partial\Omega}(AF_hx)}{h^{\frac{1}{2+\alpha}}d_{\partial\Omega_h}(x)}\right]_{C^{0,1}} \le Ch^{\frac{1}{2+\alpha}} \qquad in \quad \Omega_h \cap S_1(U_0).$$

We remark that the ratio above tends to 1 as $x \to 0$.

Proof. (i) The $C^{1,1}$ norms corresponding to Ω and $A^{-1}\Omega$ are comparable, since A is bounded. Now the conclusion follows after we apply the linear transformation F_h^{-1} to $A^{-1}\Omega$. One way to see this is to think of F_h^{-1} as a composition of a dilation of factor $h^{-\frac{1}{2}}$ followed by a $h^{\frac{1}{2}-\frac{1}{2+\alpha}}$ contraction along the x_n direction.

(ii) The Lipschitz estimate follows from Lemma 6.2 below which estimates the Lipschitz norm of the ratio of two positive $C^{1,1}$ functions vanishing on the boundary. We apply Lemma 6.2 to Ω_h, $v(x) = d_{\partial\Omega_h}(x)$, $u(x) = h^{-\frac{1}{2+\alpha}}d_{\partial\Omega}(X)$ with $X = AF_hx$, and $\varepsilon = Ch^{\frac{1}{2+\alpha}}$ to obtain the desired conclusion.

Indeed, by (i), the curvatures of $\partial\Omega_h \cap S_1(U_0)$ are bounded by $Ch^{\frac{1+\alpha}{2+\alpha}} \le \varepsilon$, thus $|D^2v| \le C\varepsilon$. Furthermore, since $|D_X^2 d_{\partial\Omega}| \le C$, we obtain

$$|D^2u(x)| = h^{-\frac{1}{2+\alpha}}|F_h^T A^T D_X^2 d_{\partial\Omega} AF_h| \le Ch^{\frac{1}{2+\alpha}}.$$

The proof of (ii) is complete. $\square$

Below we present a general Lipschitz bound for the quotient of two positive $C^{1,1}$ functions vanishing continuously on the boundary os some $C^{1,1}$ domain.

Lemma 6.2. *Assume that Ω is a $C^{1,1}$ domain with norm bounded by ε, and suppose $0 \in \partial\Omega$. Let u, v be positive $C^{1,1}$ functions in $\Omega \cap B_1$ that vanish continuously on $\partial\Omega \cap B_1$ and assume $v_\nu \ge c$ on $\partial\Omega \cap B_1$, and in $\Omega \cap B_1$ we have*

$$|\nabla u| \le C, \quad |\nabla v| \le C, \quad [\nabla u]_{C^{0,1}} \le \varepsilon, \quad [\nabla v]_{C^{0,1}} \le \varepsilon.$$

Then

$$\left[\frac{u}{v}\right]_{C^{0,1}} \le C\varepsilon \ in \ \Omega \cap B_{1/2}.$$

Proof. For $x \in \Omega \cap B_{1/2}$, let $\xi_x = \nabla d_{\partial\Omega}(x)$ be the unit vector at x which gives the perpendicular direction to $\partial\Omega$. Then

$$\frac{u(x)}{d_{\partial\Omega}(x)} = \frac{1}{d_{\partial\Omega}(x)}\int_1^0 \left[\frac{d}{dt}u(x - td_{\partial\Omega}(x)\xi_x)\right]dt = \int_0^1 \nabla u(x - td_{\partial\Omega}(x)\xi_x) \cdot \xi_x\, dt.$$

The bound on the curvatures of $\partial\Omega$ implies $[\xi]_{C^{0,1}} \leq \varepsilon$. This combined with the hypotheses on ∇u easily gives

$$\left[\frac{u}{d_{\partial\Omega}}\right]_{C^{0,1}} \leq C\varepsilon \quad \text{in} \quad \Omega \cap B_{1/2}.$$

The same holds for v, and since $v_\nu \geq c$ on $\partial\Omega$ we find

$$\left[\frac{d_{\partial\Omega}}{v}\right]_{C^{0,1}} \leq C\varepsilon.$$

and the conclusion of the lemma follows. $\qquad\square$

Proof of Lemma 2.5. First, we consider the case when $D^2_{x'}\varphi(0) = I'$, and $g(0) = 1$. Then we can apply the pointwise C^2 estimate, Theorem 2.4 in [S2]. The theorem states that there exists a sliding

$$A\,x = x + \tau x_n, \quad \text{with } |\tau| \leq C(\rho, \rho')$$

such that for any $\eta > 0$, we have

$$(1-\eta)A\ S_h(U_0) \subset S_h(u) \subset (1+\eta)A\ S_h(U_0).$$

for all $h \leq \bar{c}(\eta)$. Thus $u_h(x) = h^{-1}u(AF_h x)$ satisfies

$$|u_h - U_0| \leq C\eta \text{ in } S_1(U_0) \cap \Omega_h$$

for some C universal. By choosing $\eta = \varepsilon_0/C$, we obtain for all $h \leq \bar{c}(\varepsilon_0)$

$$|u_h - U_0| \leq \varepsilon_0 \text{ in } S_1(U_0) \cap \Omega_h.$$

Next, we check that q_h and the boundary data φ_h satisfy on $S_1(U_0) \cap \partial\Omega_h$

(6.1) $$[D^2\varphi_h]_{C^\beta} \leq Ch^{\frac{\beta}{2}}[D^2\varphi]_{C^\beta}, \qquad \|q_h\|_{C^{2,\beta}} \leq Ch^{\frac{1+\alpha}{2+\alpha}}\|q\|_{C^{2,\beta}},$$

and hence they can be made smaller than ε_0 if we choose h small.

Let $x \in \partial\Omega_h \cap S_1(U_0)$, and let $y = AF_h\,x \in \partial\Omega$ and $z = F_h x$. Thus

$$h^{\frac{1}{2+\alpha}}x_n = z_n, \qquad h^{\frac{1}{2}}x' = z', \qquad y = Az = z + \tau z_n.$$

Let $\bar{\varphi}(z') = \varphi(y')$ and $\bar{q}(z') = q(y')$ denote the corresponding functions written in the z variables. Since A, A^{-1} are bounded linear transformations we find that the $C^{2,\beta}$ norms of $\bar{\varphi}$ and φ, respectively $\bar{q}$ and q, are comparable.

On the other hand we have

$$\varphi_h(x') = \frac{1}{h}\bar{\varphi}(z'), \quad q_h(x') = h^{-\frac{1}{2+\alpha}}\bar{q}(z').$$

and (6.1) easily follows since $z' = h^{\frac{1}{2}}x'$.

Finally, we verify the behavior of g after the rescaling AF_h.

Let $x \in \Omega_h \cap S_1(U_0)$, and let $y := AF_h x \in \Omega$. Then

$$\det D^2 u_h(x) = h^{\frac{2}{2+\alpha}-1}\det D^2u(y) = g(y)\,d^\alpha_{\partial\Omega}(y)\,h^{-\frac{\alpha}{2+\alpha}} := g_h(x)d^\alpha_{\partial\Omega_h}(x),$$

where

$$g_h(x) = g(y)\left(\frac{d_{\partial\Omega}(AF_h x)}{h^{\frac{1}{2+\alpha}}d_{\partial\Omega_h}(x)}\right)^\alpha.$$

Now we may apply Lemma 6.1 together with $|D_x y| = |AF_h| \leq Ch^{\frac{1}{2+\alpha}}$ and obtain

$$g_h(0) = g(0) = 1, \qquad [g_h]_{C^\gamma} \leq Ch^{\frac{\beta}{2}}.$$

In the general case, we perform an initial linear transformation given by a symmetric matrix D which has e_n as an eigenvector such that

$$\tilde{u}(x) = u(Dx), \quad x \in \tilde{\Omega} := D^{-1}\Omega,$$

satisfies $D^2_{x'}\tilde{\varphi} = I'$ and $\tilde{g}(0) = 0$. Then, from the computations above, it is easy to see that the norms of $\tilde{\varphi}$, $\tilde{q}$ and $\tilde{g}$ corresponding to $\tilde{u}$ are comparable to the ones of φ, q and g respectively. □

6.2. The linearized equation.

We give below the proof of Lemma 2.7 which is an application of the maximum principle and Taylor's formula. First we prove that

$$(6.2) \qquad\qquad |w| \leq C x_n \quad \text{in} \quad B^+_{1/2}.$$

Indeed, for example at $x_0 = 0 \in \{x_n = 0\} \cap B_{1/2}$, we consider the upper barrier

$$\bar{w}(x) = C x_n + 4|x'|^2 - 8n x_n^{2+\alpha},$$

with C large depending only on n such that

$$(C/2)x_n - 8n x_n^{2+\alpha} \geq 0 \text{ on } B^+_1.$$

Then $L(\bar{w}) \leq 0$, and on $\partial B^+_1 \cap \{x_n > 0\}$, we have $\bar{w} \geq 1 \geq w$. The maximum principle gives $w \leq \bar{w}$, and hence $w(0, x_n) \leq C x_n$. We obtain the desired bound by translating the function $\bar{w}$ at other points on $\{x_n = 0\} \cap B_{1/2}$.

Clearly

$$\|w\|_{C^{\frac{2}{2+\alpha}}} \leq C\|w\|_{L^\infty(B^+_1)} \quad \text{in} \quad B = B_{1/4}(1/2e_n).$$

Now we apply this estimate for the rescalings $w_h(x) = w(F_h x)$, use the bound (6.2), and obtain

$$\|w\|_{C^{\frac{2}{2+\alpha}}} \leq C \quad \text{in} \quad F_h(B), \quad \forall h \in (0,1),$$

which easily implies a bound on the norm of w in $C^{\frac{2}{2+\alpha}}(\overline{B}^+_{1/2})$.

The equation is invariant under translation in the x_i direction for all $1 \leq i \leq n-1$. Thus by iterating this argument (using first difference quotients) we find that the derivatives of w with respect to x_i of any order are bounded in $B^+_{1/2}$.

The equation $w_{nn} + x_n^\alpha \sum_{j=1}^{n-1} w_{jj} = 0$ and the bound $|w_{jj}| \leq C$ imply the bound $|w_{nn}| \leq C x_n^\alpha$. Thus, combining with the bound of w_n on $\{x_n = 0\} \cap B^+_{1/2}$, we see that w_n is bounded in $B^+_{1/2}$. The same estimates as above show that w_{in}, w_{iin} are bounded as well.

By Taylor's formula, namely

$$f(t) = f(0) + f'(0)t + \int_0^t (t-s)f''(s)ds$$

and the equation $Lw = 0$, we conclude that

$$w(0, x_n) = w(0) + w_n(0)\,x_n - \frac{\sum_{j=1}^{n-1} w_{jj}(0)}{(\alpha+2)(\alpha+1)}x_n^{2+\alpha} + O(x_n^{3+\alpha}),$$

$$w_i(0, x_n) = w_i(0) + w_{in}(0)\,x_n + O(x_n^{2+\alpha}), \qquad i < n,$$

and

$$w_{ij}(0, x_n) = w_{ij}(0) + O(x_n), \qquad i,j < n.$$

Using these identities in the following Taylor's formula for $f(x') = w(x', x_n)$

$$\begin{aligned}
f(x') &= f(0) + \sum_{i=1}^{n-1} f_i(0)x_i + \frac{1}{2}\sum_{i,j=1}^{n-1} f_{ij}(0)x_i x_j + O(|x'|^3) \\
&= w(0, x_n) + \sum_{i=1}^{n-1} w_i(0, x_n)x_i + \frac{1}{2}\sum_{i,j=1}^{n-1} w_{ij}(0, x_n)x_i x_j + O(|x'|^3)
\end{aligned}$$

we obtain the desired conclusion since $w_i(0) = w_{ij}(0) = 0$ for all $1 \leq i, j \leq n-1$. □

6.3. **Proof of Proposition 5.3.** After a dilation we may assume that v solves

$$(6.3) \qquad x_n^\alpha \, a^{ij} v_{ij} + v_{nn} = x_n^\alpha f(x) \quad \text{in} \quad S_1(U_0),$$

with $(a^{ij})_{i,j \leq n-1}$ uniformly elliptic:

$$\lambda I' \leq (a^{ij}(x))_{i,j} \leq \Lambda I',$$

and $a^{ij}, f \in C_\alpha^\gamma(S_1(U_0))$ and $v(x', 0) = \varphi(x') \in C^{2,\gamma}$.

As we remarked before, the distance d_α from Definition 5.2 occurs naturally in equation (6.3) since they both scale homogeneously after the transformations $x \mapsto F_h x$. Precisely,

$$\tilde{v}(x) = h^{-1} v(F_h x),$$

solves

$$x_n^\alpha \, \tilde{a}^{ij} \tilde{v}_{ij} + \tilde{v}_{nn} = x_n^\alpha \tilde{f}(x),$$

with

$$\tilde{a}^{ij}(x) = a^{ij}(F_h x), \quad \tilde{f}(x) = f(F_h x),$$

and

$$d_\alpha(y, z) = h^{-\frac{1}{2}} d_\alpha(F_h y, F_h z).$$

First we state a pointwise $C^{2,\gamma}$ estimate at the origin.

Lemma 6.3. *If v satisfies (6.3) as above, then there exists a tangent "polynomial" P_0 at 0,*

$$P_0(x) = q(x') + (a_0 + a' \cdot x') \, x_n + \frac{b_0}{(1+\alpha)(2+\alpha)} x_n^{2+\alpha},$$

with q a quadratic polynomial in x', and

$$a^{ij}(0) \, q_{ij} + b_0 = f(0), \qquad \|q\|, |a'|, |a_0|, |b_0| \leq C,$$

such that

$$|v - P| \leq C U_0^{1+\frac{\gamma}{2}} \quad \text{in} \quad S_{1/2}(U_0).$$

The constant C above depends on n, λ, Λ, α, γ, $\|v\|_{L^\infty}$ and the corresponding norms of a^{ij}, f, φ.

Proposition 5.3 follows from Lemma 6.3 by rescaling. Indeed, the function

$$w(x) := h^{-1-\frac{\gamma}{2}}(v - P_0)(F_h x)$$

satisfies

$$x_n^\alpha \, \tilde{a}^{ij} w_{ij} + w_{nn} = x_n^\alpha \tilde{f}(x) \quad \text{in} \quad S_1(U_0)$$

with

$$\tilde{a}^{ij}(x) = a^{ij}(F_h x), \quad \tilde{f}(x) := h^{-\frac{\gamma}{2}} \left(f(F_h x) - f(0) - (a^{ij}(F_h x) - a^{ij}(0)) q_{ij} \right).$$

Since $h^{\frac{1}{2}} |y - z| \sim d_\alpha(F_h y, F_h z)$ for $y, z \in B_c(\frac{1}{2} e_n)$, we see that

$$\|\tilde{a}^{ij}\|_{C^\gamma}, \|\tilde{f}\|_{C^\gamma} \leq C \quad \text{in} \quad B := B_{1/4}(e_n/2).$$

By Lemma 6.3 $\|w\|_{L^\infty} \leq C$, thus by the classical Schauder estimates we find

$$\|D^2 w\|_{C^\gamma} \leq C \quad \text{in} \quad B.$$

Rescaling back and using that $D^2 P \in C_\alpha^\gamma$, $D^2 v(0) = D^2 P(0)$, we find

$$\|D^2 v - D^2 v(0)\|_{L^\infty} \leq C h^{\frac{\gamma}{2}}, \quad \|D^2 v\|_{C_\alpha^\gamma} \leq C \quad \text{in} \quad F_h(B),$$

and now it is straightforward to check the conclusion of Proposition 5.3.

Lemma 6.3 is standard and we only sketch its proof below.

Proof of Lemma 6.3. After subtracting the boundary data and a multiple of $x_n^{2+\alpha}$, and then after a suitable linear transformation, a dilation and multiplication by a constant we may assume that in $S_1(U_0)$, $v = 0$ on $\{x_n = 0\}$, $\|v\|_{L^\infty} \le 1$, and

$$a^{ij}(0) = \delta_{ij}, \quad [a^{ij}]_{C_\alpha^\gamma} \le \delta, \quad f(0) = 0, \quad [f]_{C_\alpha^\gamma} \le \delta,$$

for some sufficiently small $\delta > 0$. As before, we use barriers for v near $x_n = 0$ and conclude by compactness that, as $\delta \to 0$, v can be approximated uniformly in $S_{1/2}(U_0)$ by a solution to

$$x_n^\alpha \Delta_{x'} \bar{v} + \bar{v}_{nn} = 0, \qquad \bar{v} = 0 \quad \text{on} \quad \{x_n = 0\}, \qquad \|\bar{v}\|_{L^\infty} \le 1,$$

i.e.

$$v = \bar{v} + o(1) \quad \text{in} \quad S_{1/2}(U_0) \quad \text{where} \quad o(1) \to 0 \quad \text{as} \quad \delta \to 0.$$

By Lemma 2.7, there exists θ_0 small depending only on n, α such that

$$|\bar{v} - \bar{P}| \le C_0 \theta_0^{\frac{3+\alpha}{2+\alpha}} \le \frac{1}{2}\,\theta_0^{1+\frac{\gamma}{2}} \quad \text{in} \quad S_{\theta_0}(U_0)$$

for some quadratic polynomial

$$\bar{P}(x) := (\bar{a}_0 + \bar{a}' \cdot x')x_n.$$

In conclusion

$$|v - \bar{P}| \le \theta_0^{1+\frac{\gamma}{2}} \quad \text{in} \quad S_{\theta_0}(U_0),$$

provided that δ is chosen sufficiently small.

Now the rescaling of $v - \bar{P}$ from $S_{\theta_0}(U_0)$ to $S_1(U_0)$ satisfies the same hypotheses as v above. Then we can iterate the same argument and obtain the desired conclusion. $\qquad\square$

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